OVERVIEW

In the *Population Dynamics* Click & Learn, students explore two classic mathematical models that describe how populations change over time: the exponential and logistic growth models. Students learn about each model through an interactive simulator supported by introductory information and real biological examples. The accompanying "Student Worksheet" guides students' exploration by having them analyze different components of the models, generate and interpret related plots, and investigate examples of population growth in bacteria and humans. The "African Wildlife Case Studies" handout allows students to apply the *Population Dynamics* Click & Learn to three case studies involving African antelope populations, including wildebeest.

Additional information related to pedagogy and implementation can be found on this resource's webpage, including suggested audience, estimated time, and curriculum connections.

KEY CONCEPTS

- Mathematical population models can be used to describe and simulate common patterns of population growth.
- The exponential growth model describes how a population grows when it has unlimited resources. In this model, the population continues growing larger and faster over time.
- The logistic growth model describes how a population grows when it is limited by resources or other density-dependent factors. In this model, the population grows more slowly as it approaches a limit called the carrying capacity.
- The results of these models can be plotted to simulate and analyze a population's projected trajectory over time.

STUDENT LEARNING TARGETS

- Describe the assumptions of the exponential and logistic growth models, and how those assumptions do or do not apply to different populations.
- Interpret and apply the exponential and logistic growth equations.
- Explain how the key variables and parameters in these models — such as time, the maximum per capita growth rate, the initial population size, and the carrying capacity — affect population growth.
- Use the exponential and logistic growth models to project and interpret real biological examples.
- Interpret and analyze plots summarizing population growth over time.

PRIOR KNOWLEDGE

Students should be familiar with:

- basic ecological concepts of populations and population growth
- the general uses, strengths, and limitations of mathematical models
- interpreting and applying mathematical equations
- interpreting patterns on plots

BACKGROUND

The "Introduction" sections in the *Population Dynamics* Click & Learn provide general background on exponential and logistic growth. The Click & Learn also briefly covers the differences between continuous-time and discrete-time population models and includes examples of how continuous-time models in particular can be applied to bacteria.

The section below and the appendices at the end of this document provide additional, more advanced information about the mathematics of these models. These parts of the document are *optional* and not required for doing the Click & Learn.

Continuous-time vs. discrete-time models

Although the *Population Dynamics* Click & Learn focuses on continuous-time models, the "Introduction" sections include some supplemental information on discrete-time models. As mentioned in these sections, **continuous-time models** describe populations that are changing all the time. They are often used for organisms that reproduce year-round, such as bacteria or humans. **Discrete-time models** describe populations that change mainly over specific time periods. They are often used for organisms that reproduce or have high mortality rates seasonally, such as many insects and annual plants. If you make the time periods in a discrete-time model infinitely small ($\Delta t \rightarrow 0$), it essentially becomes a continuous-time model.

As also noted in the Click & Learn, the **maximum per capita growth rate** in the *continuous*-time model (written there as r; also known as the instantaneous rate of increase, intrinsic growth rate, intrinsic capacity for increase, etc.) is slightly different from the maximum per capita growth rate in an equivalent *discrete*-time model (written there as r_d; also known as the discrete growth factor). These rates are related as shown in the following equation:

$$r = \ln(r_d + 1)$$

A derivation of this result is as follows. In a *continuous*-time exponential model, the population size N at time t is given by the following equation. (See Appendix 1 at the end of this document for more details.):

$$N(t) = N_0 e^{rt}$$

In a *discrete*-time exponential model, the population size N_t at time t is given by the following equation. (See the "The discrete-time exponential growth model" section in the Click & Learn for more details.):

$$N_t = N_0(r_d + 1)^t$$

If the models are equivalent, $N(t)$ in the continuous-time model will equal N_t in the discrete-time model. So:

$$N_0 e^{rt} = N_0(r_d + 1)^t$$

Simplify the equation above to obtain the relationship between r and r_d:

$$e^r = r_d + 1$$
$$r = \ln(r_d + 1)$$

You will often see $r_d + 1$ written as λ (lambda), which is known as the **finite rate of increase**. So:

$$e^r = \lambda$$
$$r = \ln(\lambda)$$

TEACHING TIPS

- It may be helpful to discuss the settings and plots for the simulators in the *Population Dynamics* Click & Learn as a class before students start using them.
  - **Plot 1** shows the population size (N) as a function of time (t). It includes the ability to display the population growth rate (dN/dt) as the slope of this curve at a given point.
    - You may need to explain to students why dN/dt is the slope. In calculus, dN/dt is the *derivative* of N with respect to t. This means that dN/dt at time t is the *slope* of $N(t)$ at that time.
  - **Plot 2** shows the population growth rate (dN/dt) as a function of the population size (N).
    - Note that Plot 2 uses only the values of N that appear on Plot 1. You may need to adjust N_0 or the "Min" and "Max" for N on Plot 1 in order to see more values on Plot 2.
    - You may need to remind students that N is a function of time t. So, although t does not appear on Plot 2 directly, it determines the values of N that are shown.

  - The **"Help"** tab on the bottom of the Click & Learn provides important information about settings and limitations for the simulators.
    - In particular, the simulators may behave suboptimally at very large values of the variables and/or settings. This may result in lag or visual glitches, such as incorrect fluctuations in the curves. You may want to double-check such results with another graphing software.
- There are two student documents that can be optionally used with the *Population Dynamics* Click & Learn. Please scaffold or modify these documents as needed (e.g., reduce the number of questions, simplify questions, etc.) in order to better fit your learning objectives and your students' needs.
  - The **"Student Worksheet"** provides a general introduction to the characteristics of exponential and logistic growth models and the simulators in the Click & Learn.
    - Question 6 asks students to provide units for the parameters in the model. Students may be confused why the units of the maximum per capita growth rate (*r*) are "per time" (e.g., "per hour" or "per year") rather than "individuals per time." You may need to explain that the units of the overall population growth rate are "individuals per time," but the units of a *per capita* growth rate are "individuals per time *per individual*," which reduces to just "per time."
    - Question 11 asks students to identify specific combinations of parameter values that produce certain patterns. This is a good opportunity for a class discussion where students can share their different answers and try to determine general ranges of parameters for each pattern.
    - Question 12 and the optional question after Question 19 ask students to perform calculations with exponents. You may have students skip these questions if they are unfamiliar with exponents; instruct them or edit the documents accordingly. Note that the optional question involves doing calculations with the logistic growth equation, which may be overly complex for some students.
    - Several questions (such as Questions 9, 10, 14, etc.) ask students to qualitatively describe or draw patterns they observe in the plots. You could also consider having students record specific values from the simulator and make claims based on the values they collected.
    - Several questions (such as Questions 14, 21, and 25) ask students to sketch graphs. If students are doing the worksheet online, they could upload digital drawings, take photos of sketches drawn on paper, or write descriptions of the graphs instead.
  - The **"African Wildlife Case Studies"** handout has students apply the models and simulators in the Click & Learn to three case studies involving African antelope populations.
    - If students are not familiar with the animals in these case studies (waterbuck, kudu, and wildebeest), consider showing them images or videos of the animals before doing the activity.
    - If the antelope examples are not as relevant or engaging to your students, consider adapting the handout to create your own case studies based on local examples, students' interests, other topics they are learning about in class, etc.
    - Figure 1 is based on data from the following references:
      - Mduma, Simon A. R., A. R. E. Sinclair, and Ray Hilborn. "Food regulates the Serengeti wildebeest: A 40-year record." *Journal of Animal Ecology* 68, 6 (1999): 1101–1122. https://doi.org/10.1046/j.1365-2656.1999.00352.x.
      - Grange, Sophie, et al. "What limits the Serengeti zebra population?" *Oecologia* 140, 3 (2004): 523–532. https://doi.org/10.1007/s00442-004-1567-6.
- The examples in the "African Wildlife Case Studies" handout can be supplemented with related BioInteractive resources, such as the following:

  - The **waterbuck** case study is based on the 2015 Holiday Lecture "Modeling Populations and Species Interactions" by mathematical biologist Corina Tarnita. The lecture provides more context for the waterbuck example and explains how some of the parameters in the model were estimated. Other BioInteractive resources, such as the interactive *Gorongosa Timeline*, can be used to give more background on Gorongosa National Park, the site of the case study.
  - The **wildebeest and rinderpest** case study is explored in multiple BioInteractive resources, including the Scientists at Work video *Mystery of the Buffalo Boom*, the short film *Serengeti: Nature's Living Laboratory*, and the Data Point activity "Serengeti Wildebeest Population Regulation."
- The Click & Learn mentions that calculus can be used on the equations for *dN/dt*, in both the continuous-time exponential and logistic models, to find equations for *N* as a function of *t*. Derivations of these equations are included in the two appendices *at the end of this document* and can be optionally provided to students with knowledge of algebra and calculus in order to practice their math skills.
  - **Appendix 1** shows how to derive an equation for $N(t)$ in the *exponential* growth model. You may wish to share it with students who are familiar with first-order differential equations.
  - **Appendix 2** shows how to derive an equation for $N(t)$ in the *logistic* growth model. This derivation is more challenging than the one for the exponential growth model but may be of interest to students with a strong background in calculus and algebra.
- Discrete-time exponential and logistic models are briefly discussed in the "Introduction" sections of the Click & Learn but not in the "Student Worksheet." You may wish to provide students with additional materials on discrete-time models if you are interested in covering them.
- The simulators in the Click & Learn include an option to toggle between logarithmic and linear scales for the *y*-axes (population size and population growth rate).
  - Logarithmic scales are not explicitly covered in the student documents. If you are interested in covering logarithmic scales, consider asking students to toggle between the "Linear" and "Log" scales for some of the worksheet questions. They can record how switching scales affects the patterns of growth that they observe. For example, they could switch between scales as they observe the effects of r or N_0 on Plot 1 or Plot 2.
  - If students are unfamiliar with logarithms or interpreting logarithmic scales, consider discussing them as a class and/or providing additional support.

ANSWER KEY: STUDENT WORKSHEET

PART 1: Introduction to Population Dynamics

1. Describe a specific question or problem related to population dynamics that interests you.
   Student answers will vary. The Click & Learn gives several broad examples of how population dynamics can be related to conservation, pest and disease control, agriculture, and human populations.
2. Do you think the question or problem you described could be investigated using a mathematical population model? Why or why not?
   Student answers will vary. You may want to emphasize that a tremendous variety of questions and problems can be investigated using mathematical models, and that the Click & Learn shows only a few examples of what models can do.

PART 2: Exploring the Exponential Growth Model

3. The end of the "Introduction" describes how you could use a continuous-time, exponential growth model to simulate an *E. coli* population growing in a lab. Describe *another* specific population and situation that you

could simulate with this type of model.

Student answers will vary. For a continuous-time model, they should pick a population that is changing all the time due to year-round births or deaths. For exponential growth, their population should also not be limited by food, space, or other density-dependent factors.

4. Complete the following table to explain the *biological* meanings of the symbols in the exponential growth model. For each explanation, give a specific example using the population you described above.
Answers will vary depending on students' interpretations and the populations that they chose. Example answers for a population of bacteria are shown below.

Symbol	Biological Meaning	Specific Example
N	*the number of individuals in the population*	*how many bacteria are in the population at a given time*
t	*time*	*whenever we are measuring the bacteria*
dN/dt	*the overall rate of change in the population size*	*how quickly the population is gaining or losing bacteria*
r	*the largest possible growth rate of the population, per individual*	*how quickly the population can gain bacteria when it's growing as fast as possible, measured per bacteria*
N_0	*the initial population size*	*how many bacteria were in the population when we first started measuring it*

5. Both dN/dt and r are types of growth rates. What are the differences between them?
dN/dt *is the overall population growth rate, which describes how fast the population is actually growing at a given time.* r *is the maximum per capita growth rate, which is a constant that describes how fast the population could grow if it's growing as fast as possible.*

In these models, dN/dt *changes over time (depends on t), but* r *stays the same (does not depend on* t*). Also,* r *is a per capita growth rate, meaning that it's measured per individual, whereas* dN/dt *is measured for the overall population.*

6. No units are shown for the numbers in the "Settings" section. This is because each of these numbers can have many possible units. Give an example of possible units for each of the following:
Student answers will vary. Several examples are shown below.
   a. N_0
   individuals, animals, bacteria, etc.
   b. t
   seconds, hours, years, etc.
   c. r, using the units for t you gave above
   per second (equivalently, 1/seconds or seconds^{-1}), where "second" is replaced with whatever time unit was chosen in Part B. These units could also be written in the form of "individuals per second, per individual," which reduces to "per second" as well.

   Students may be confused why the units of the maximum per capita growth rate (r) are "per time" (e.g., "per second") rather than "individuals per time." You may need to explain that the units of the overall population growth rate are "individuals per time," but the units of a per capita *growth rate are "individuals per time* per individual*," which reduces to just "per time."*

7. Examine both Plot 1 and Plot 2.
   a. In Plot 1, what variables do the x- and y-axes represent?
   The x*-axis represents time (*t*) and the* y*-axis represents population size (*N*).*

b. In Plot 2, what variables do the *x*- and *y*-axes represent?
The x*-axis represents population size (*N*) and the* y*-axis represents population growth rate (*dN/dt*).*

8. Set N_0 = 50, *r* = 0.5, and *t* = 5.

   a. What is the population size at this point?
   *The population size (*N*) is 609.*

   b. What is the population growth rate at this point?
   *The population growth rate (*dN/dt*) is 304.56.*

9. Set *r* = 0.1, then gradually increase *r* by clicking the up-arrow to the right of the number. You may need to hover over the number to see the arrow.

   a. Examine Plot 1. As you increase *r*, what happens to the curve of population size over time?
   As r *increases, this curve becomes steeper more quickly, showing that the population size increases more rapidly.*

   b. Examine Plot 2. As you increase *r*, what happens to the curve of population growth rate vs. population size?
   *This curve is always a straight line, which means that the population growth rate (*dN/dt*) increases linearly with population size (*N*). But as* r *increases, the line on Plot 2 becomes steeper, which indicates that the population growth rate is increasing more rapidly with N.*

10. Set *r* = 0.5 and N_0 = 5, then gradually increase N_0 by clicking the up-arrow to the right of the number.

   a. Examine Plot 1. How does the curve of population size over time change if you start with a smaller number of individuals (e.g., N_0 = 5) compared to a larger number of individuals (e.g., N_0 = 100)?
   When the population starts with a smaller number of individuals, the curve of population size over time is almost flat at the start, meaning that the population's growth is initially slow. Over time, the curve increases more rapidly, meaning that the population is growing faster. When the population starts with a larger number of individuals, the curve rises very quickly from the start, meaning that the population is growing quickly from the start.

   b. Examine Plot 2. How does the curve of population growth rate vs. population size change if you start with a smaller number of individuals compared to a larger number?
   This curve does not change when you change the initial population size. (Note that Plot 2 uses only the values of N *that appear in Plot 1, so students may see more or less of this curve depending on the values of* N *in their Plot 1. However, the curve itself will not change.) This means that the initial population size (*N_0*) does not affect the relationship between the population growth rate (*dN/dt*) and population size (*N*).*

11. List one combination of values for *r* and N_0 that produces each of the following patterns for population size over time. (There are many possible answers.) Use a time range with a "Min" of 0 and a "Max" of 10.
Many answers are possible; some examples are shown below. Consider discussing students' responses as a class to come up with general ranges of parameters for each pattern.

Pattern	Value of *r*	Value of N_0
A long period of what appears to be almost no growth. (The curve in Plot 1 looks almost flat.)	*Small (e.g., 0.2)*	*Small (e.g., 1)*
A long period of slow but clearly accelerating growth. (The curve in Plot 1 starts to become steeper at the end.)	*Intermediate (e.g., 0.5)*	*Small (e.g., 1)*
Extremely fast growth from the very beginning.	*Large (e.g., 1)*	*Large (e.g., 100)*

PART 3: Exponential Growth in Bacteria

12. The example claims that the population growth rate at 24 hours will be 1.69×10^{26} bacteria per hour. Confirm this result by showing your calculations below. (*Hint*: The example contains the equations and values that you'll need to use.)

This calculation is very similar to the one shown in the example, just using 24 hours instead of 1 hour. First, calculate the population size at 24 hours:

$$\begin{aligned} N(t) &= N_0 e^{rt} \\ &= (2 \text{ bacteria})\left(e^{2.45 \text{ per hour} \times 24 \text{ hours}}\right) \\ &= (2)(e^{2.45 \times 24}) \text{ bacteria} \\ &= 6.88 \times 10^{25} \text{ bacteria} \end{aligned}$$

Use this value of N *to find the population growth rate at 24 hours:*

$$\begin{aligned} \frac{dN}{dt} &= rN \\ &= (2.45 \text{ per hour})(6.88 \times 10^{25} \text{ bacteria}) \\ &= 1.69 \times 10^{26} \text{ bacteria per hour} \end{aligned}$$

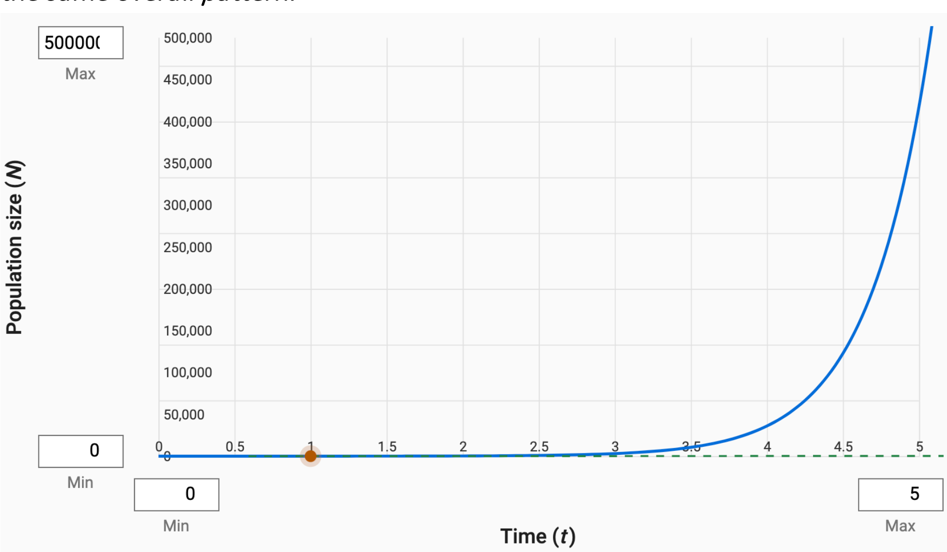
13. Using the simulator, fill in the following table with the population size (*N*) and population growth rate (*dN/dt*) at different time points (*t*, measured in hours).

Time (*t*)	1	2	3	4	5
Population size (*N*)	*23*	*269*	*3112*	*36,067*	*417,963*
Population growth rate (*dN/dt*)	*56.78*	*658.02*	*7625.36*	*88,365.35*	*1,024,008.32*

14. Use your table above and/or the simulator to answer the following questions. (*Hint:* For the simulator, you may want to change the "Max" values for the axes on Plot 1 to get a better look at the curve. You can use the values of *t* and *N* from your table above to decide what the "Max" values should be.)

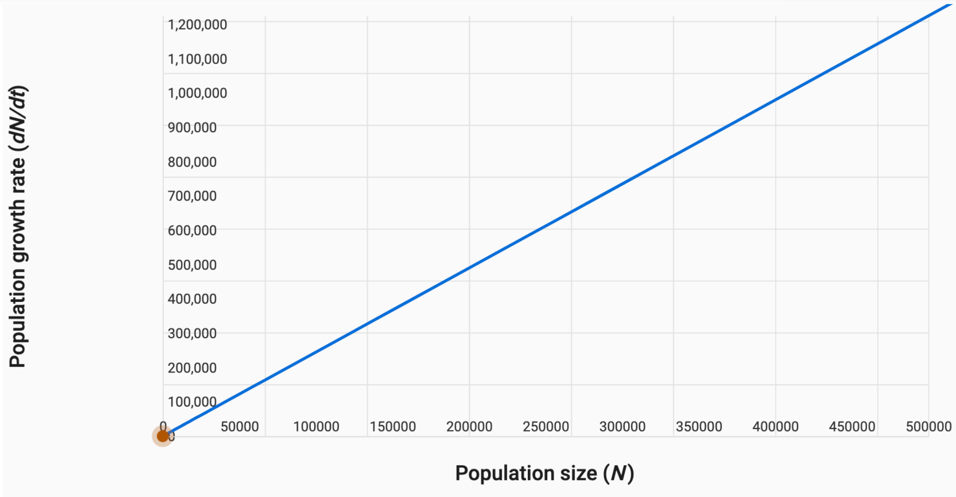
a. Sketch how the population size (*N*) changes over time.

A screenshot from the simulator is shown below. Student sketches may be less detailed but should follow the same overall pattern.



b. Sketch how the population growth rate (*dN/dt*) changes based on population size (*N*).

A screenshot from the simulator is shown below. Student sketches may be less detailed but should follow the same overall pattern.



c. The population growth rate (*dN/dt*) depends on the maximum per capita growth rate (*r*). Does *r* also change based on time or population size? Why or why not?

No, r is a constant, so it will not change with time or population size. (In the simulator, for example, you use only one value for r when making a plot.)

15. All models have strengths and limitations. A *strength* of a model could be something that the model simulates very well or something that makes it easy to use. A *limitation* could be something that the model does not simulate as well or an important process that it does not include.

a. What is one *strength* of the exponential growth model you explored?

Student answers will vary. They may say that the exponential growth model is good at simulating populations with unlimited resources, or that it is relatively simple to use or analyze compared to the logistic growth model.

b. What is one *limitation* of the exponential growth model you explored?

Student answers will vary. They may say that the exponential growth model cannot be applied to many situations, since populations rarely have unlimited resources and usually can't grow forever.

PART 4: Exploring the Logistic Growth Model

16. Summarize the main differences between the exponential and logistic growth models.

The exponential growth model describes a population with unlimited resources, which keeps growing bigger and faster over time. The logistic growth model describes a population that has limited resources or other limits to growth, which grows more slowly as it gets larger.

17. Explain what the carrying capacity (K) is in your own words.

Student answers may vary. They should generally indicate that the carrying capacity is the largest size of a population that the environment can support in the long run.

18. Set $N_0 = 1$, $r = 0.6$, and $K = 1000$. Also set the "Max" value for t on the x-axis of Plot 1 to 25.

a. Examine Plot 1. What happens to the population size over time?

The population increases in size until it reaches the carrying capacity.

b. Examine Plot 2. For what values of N is the population growth rate almost zero (for example, 0.01 or lower)?

The population growth rate is almost zero for values of N that are close to 0 or close to the carrying capacity (1000).

c. Set $N_0 = 1500$. What happens to the population size over time now? For what values of N is the population growth rate almost zero?

The population decreases in size until it reaches the carrying capacity. The population growth rate is almost zero for values of N *that are close to the carrying capacity (1000).*

d. In general, for what values of *N* and *K* is the population growth rate almost zero?
In general, the population growth rate is almost zero for values of N *that are close to 0 or close to the carrying capacity.*

19. Set N_0 = 1 again. Gradually increase *r* by clicking the up-arrow on its box.

a. Examine Plot 1. As you increase *r*, what happens to the curve of population size over time?
The population increases more quickly at the beginning and reaches the carrying capacity faster.

b. Examine Plot 2. As you increase *r*, what happens to the curve of population growth rate vs. population size? (*Hint:* Pay attention to the numbers on the *y*-axis of Plot 2.)
In general, the population growth rate is low when N *is close to 0 or close to the carrying capacity (1000), but it has a peak between these two extremes. As* r *increases, the range on the* y*-axis of Plot 2 also increases, indicating that the peak is getting taller. (It can be shown that the maximum population growth rate occurs at* N = K*/2 and has a value of* dN/dt = rK*/4. This is why the peak of the curve on Plot 2 increases with* r*.)*

PART 5: Logistic Growth in Bacteria

Optional Question: The example claims that the population growth rate at 240 hours will be about 16,300 bacteria per hour. Confirm this result by showing your calculations below.
This calculation is very similar to the one shown in the example, just using 240 hours instead of 24 hours. First, calculate the population size at 240 hours:

$$\begin{aligned} N(t) &= \frac{KN_0}{N_0 + (K - N_0)e^{-rt}} \\ &= \frac{(10^{13} \text{ bacteria})(2 \text{ bacteria})}{(2 \text{ bacteria}) + (10^{13} \text{ bacteria} - 2 \text{ bacteria})e^{-(0.05 \text{ per hour})(240 \text{ hours})}} \\ &= \frac{(10^{13})(2\,)}{(2\,) + (10^{13}\; - 2\,)e^{-12}} \text{ bacteria} \\ &= 325{,}510 \text{ bacteria} \end{aligned}$$

Use this value of N *to find the population growth rate at 24 hours:*

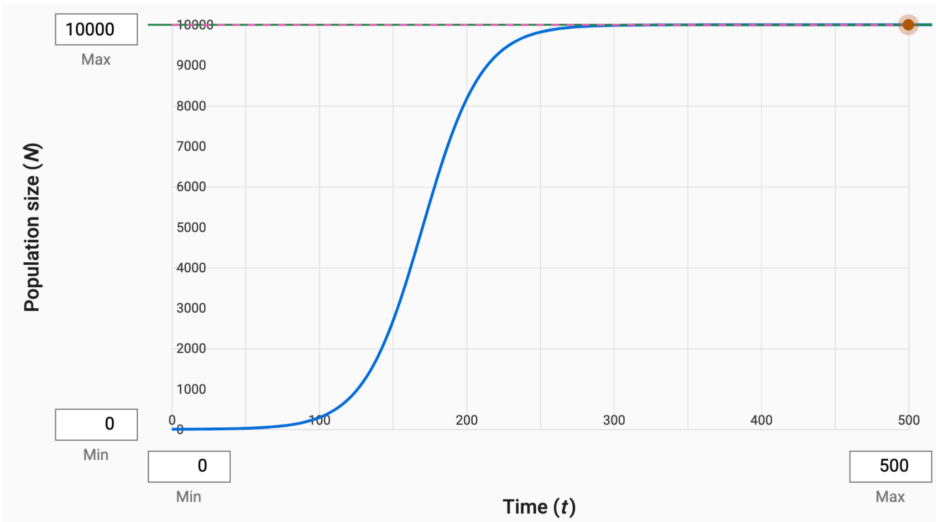
$$\begin{aligned} \frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) \\ &= (0.05 \text{ per hour})(325{,}510 \text{ bacteria})\left(1 - \frac{325{,}510 \text{ bacteria}}{10^{13} \text{ bacteria}}\right) \\ &= 16{,}275 \text{ bacteria per hour} \end{aligned}$$

20. Using the simulator, fill in the following table with the population size (*N*) and population growth rate (*dN/dt*) at different time points (*t*, measured in hours).

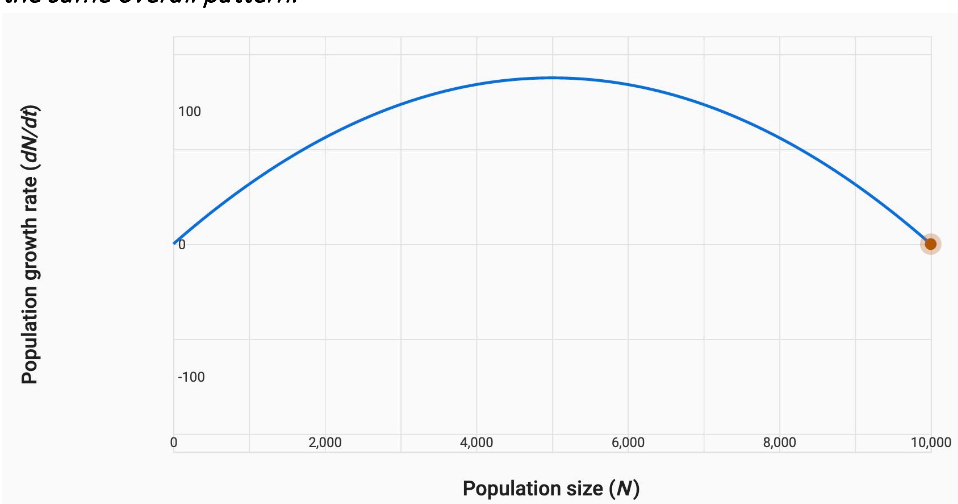
Time (*t*)	100	200	300	400	500
Population size (*N*)	*288*	*8,150*	*9,985*	*10,000*	*10,000*
Population growth rate (*dN/dt*)	*14.00*	*75.38*	*0.76*	*0.01*	*0.00*

21. Use your table above and/or the simulator to answer the following questions. (*Hint:* For the simulator, you may want to change the "Max" values for the axes on Plot 1 to get a better look at the curve. You can use the values of *t* and *N* from your table above to decide what the "Max" values should be.)

    a. Sketch how the population size (*N*) changes over time.
    A screenshot from the simulator is shown below. Student sketches may be less detailed but should follow the same overall pattern.

    b. Sketch how the population growth rate (*dN/dt*) changes based on population size (*N*).
    A screenshot from the simulator is shown below. Student sketches may be less detailed but should follow the same overall pattern.

    c. How do your answers above compare to your answers for the exponential growth model (Question 14)?
    Student answers will vary depending on their observations. They will likely notice that the population size reaches a limit (the carrying capacity) over time in the logistic growth model, but it increases without limit in the exponential growth model. The population growth rate in the exponential growth model also increases without limit, but it decreases as N nears the carrying capacity in the logistic growth model.

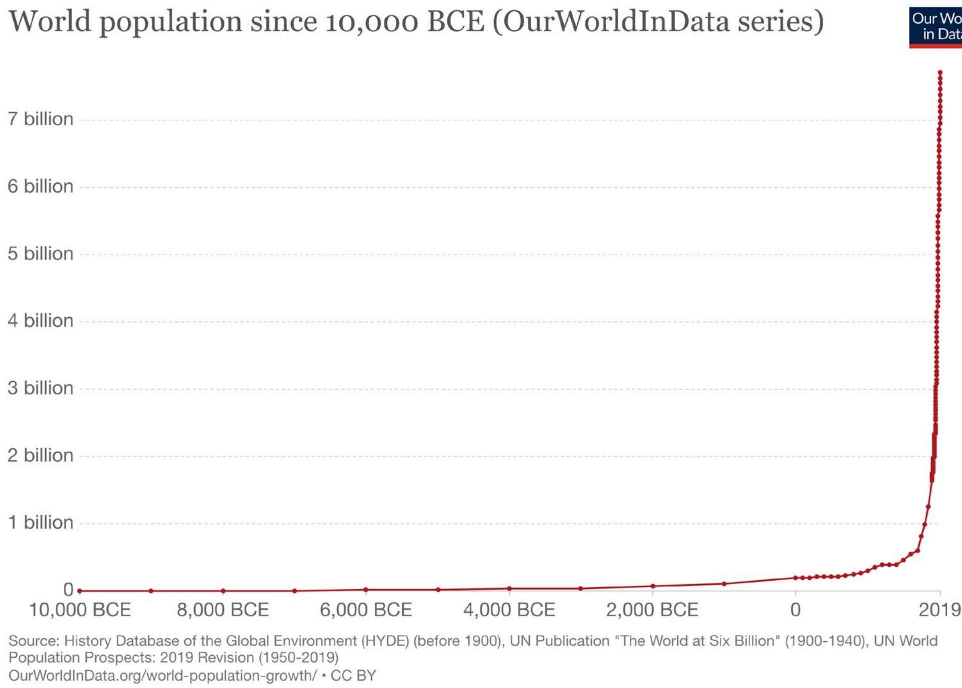
22. Complete the following table to explain why the population growth rate (*dN/dt*) is small in certain situations. The first row is filled out for you as an example.

The population growth rate (dN/dt) is small when…	Mathematical explanation	Biological explanation
the population size (N) is close to 0	The equation for the population growth rate is $dN/dt = rN(1–N/K)$. When N is close to 0, both rN and N/K are small, which makes dN/dt small too.	When the population size is small, the population has only a few individuals to produce offspring. This means the population can't grow very quickly, so the population growth rate is small.
the population size (N) is close to the carrying capacity (K)	*The equation for the population growth rate is dN/dt = rN(1–N/K). When N is close to K, the rightmost term, (1–N/K), is almost zero. This makes dN/dt almost zero (very small) too.*	*Multiple explanations involving density-dependent factors are possible. For example, when the population size is close to the carrying capacity, the population is probably using up most of the resources in its environment. Each individual won't get as many resources, so they are more likely to die or won't produce as many offspring. This makes the population grow more slowly.*

23. Like all models, the logistic growth model has both strengths and limitations.
    a. What is one *strength* of the logistic growth model you explored?
    Student answers will vary. They may say that the logistic growth model is more realistic or applies to more situations than the exponential growth model does, since many populations have limits to their growth.
    b. What is one *limitation* of the logistic growth model you explored?
    Student answers will vary. They may describe other factors they are interested in, such as age or sex structure, that were not captured by the simple model they explored. You may want to mention that there are other versions of exponential and logistic growth models that can capture these factors.
    c. How do your answers above compare to your answers for the exponential growth model (Question 15)?
    Student answers will vary depending on their previous responses.

PART 6: Modeling Other Populations

24. Do you think the global human population is experiencing exponential growth or logistic growth? Why?
    Student answers will vary. Many students may pick logistic growth because humans have limited resources (food, land, energy sources, etc.).
25. Find a graph of the global human population over time and sketch it below.
    Student answers will vary. Example graphs, such as the one shown below, can be found on the Our World in Data website.



26. Based on the graph you found, would you change your answer to Question 24? Why or why not?
Student answers will vary depending on their responses to Questions 24 and 25. Students may be surprised that the historical growth of the global human population appears to be exponential rather than logistic.

27. What do you think will happen to the size of the global human population in the long run? Why?
Student answers will vary depending on what factors they consider. They may say that it doesn't seem possible for the human population to increase forever, since it would eventually exceed Earth's available resources. Over time, growth rates may slow and better fit the pattern of logistic growth. (In fact, many countries have shifted to lower growth rates as they have become more industrialized.)

28. Think of a population from a species *not* yet discussed that you are familiar with or have learned about.

    a. Propose a specific question about this population that you could investigate using a mathematical population model.
    Student answers will vary. Be open to a range of reasonable responses.

    b. What kind of model, exponential or logistic, would you use to simulate this population and why?
    Students should generally pick exponential growth models for populations with unlimited resources/growth and logistic growth models for populations with limited resources or other density-dependent factors that limit growth (predation, disease, etc.).

    c. Is there anything you would need to add to the model you chose in Part B in order to answer your question in Part A? If so, what?
    Student answers will vary depending on the population and question they chose. Be open to a range of reasonable responses.

ANSWER KEY: AFRICAN WILDLIFE CASE STUDIES

PART 1: Waterbuck

1. How could we use mathematical models to help waterbuck and other wildlife?
There are many possible answers. For example, we could use models to project the size of wildlife populations in the future, to see which populations are in danger of dying out. We could also use models to simulate the

effects of different management actions to determine what action would be best for a population, or just to gain a better understanding of the factors affecting the population's dynamics.

2. What are the advantages of using a mathematical model to study a population rather than just observing the population?
There are many possible answers. For example, models may be preferable if observing the population directly is difficult (e.g., the animals are hard to track) or costly (e.g., the animals require a lot of time and equipment to monitor). Models can also be used to project the effects of different scenarios and conditions, and to make predictions about things that haven't happened yet or can't be tested directly.

3. At the start of the recovery period, the waterbuck population contained only 140 individuals. The population had 0.67 births per year per individual and 0.06 deaths per year per individual.

   a. What is the maximum per capita growth rate (r) for this population? Include units in your answer.
   0.61 individuals per year per individual (can also be written as just "0.61 per year")

   b. What is the initial population size (N_0) for this population? Include units in your answer.
   140 waterbuck

4. Using the simulator, fill in the following table with the population size (N) and population growth rate (dN/dt) at different time points (t, measured in years).

Time (t)	5	10	15	20	25
Population size (N)	*2,956*	*62,420*	*1,318,022*	*27,830,481*	*587,650,195*
Population growth rate (dN/dt)	*1,803.25*	*38,076.25*	*803,993.21*	*16,976,593.51*	*358,466,619.03*

5. Based on this model, how will the waterbuck population grow over time? Will the population ever stop growing or get smaller?
In this model, the waterbuck population grows bigger and faster over time. It never stops growing or gets smaller.

6. Do you think this model reflects how the waterbuck population will grow in real life? Why or why not?
In real life, the population is unlikely to grow forever like in this model. It's likely for something to eventually limit the population's growth. For example, the population could run out of food or space, or experience more disease and predation, when it gets too large.

7. Imagine that a decrease in the number of predators lowered the per capita death rate of the waterbuck to 0.04 deaths per year per individual.

   a. What would be the new maximum per capita growth rate (r) for the waterbuck population?
   0.63 per year

   b. What would be the population size (N) after 20 years (t = 20)? Use the same N_0 as in Question 3.
   41,518,199 waterbuck

8. Imagine that new waterbuck immigrate into the park at a rate of 0.25 per year. Assume that there are no emigrations and that the rest of the population parameters are the same as in Question 3.

   a. What would be the population size after 20 years (t = 20)?
   4,130,409,628 waterbuck

   b. How does the size of the population *with* immigration (your answer to Part A) compare to the size of the population *without* immigration (your result for t = 20 in Table 1)?
   The population size with immigration (4,130,409,628 waterbuck) is much bigger than the population size without immigration (27,830,481 waterbuck). Even small changes in the growth rate can have a big impact in the exponential growth model, especially at later time points.

PART 2: Kudu

9. Besides food and space, what are **two** other factors that could limit the size of a population?
There are many possible answers. Examples include other density-dependent factors, such as increased rates of predation, disease, and parasitism when the population is large. The population could also be reduced by density-independent factors such as habitat loss, pollution, or natural disasters (fire, floods, droughts, etc.).

10. What are the values of K, r, and N_0 for this kudu population?
K = 100 kudu, r = 0.26 per year, N_0 = 10 kudu

11. Based on this model, about how many years will it take the kudu population to reach the carrying capacity? (*Hint*: You may want to change the "Max" values for the axes on Plot 1 to get a better look at the curve.)
about 29 years

12. What will happen to the population growth rate (dN/dt) as the population size (N) gets closer and closer to the carrying capacity?
The growth rate will initially increase until the population size reaches about half the carrying capacity (50). The growth rate will then decrease to 0 as the population gets closer and closer to the carrying capacity (100).

13. Imagine that more land is added to the park, allowing it to support up to 250 kudu. How will the size of the kudu population change once this land is added?
The population will probably grow until it reaches the new carrying capacity of 250 kudu.

14. Reset the model to the values you determined in Question 10. Now imagine that trophy hunters start killing kudu in the park, which decreases their maximum per capita growth rate to 0.15 per year. How would this impact the kudu's population size over time? (*Hint:* Look at how many years it will take the population to reach its carrying capacity now.)
The population will grow more slowly after the hunters start killing kudu. Without the hunters, the population reached the carrying capacity in about 29 years. With the hunters, the population does not reach the carrying capacity until about 50 years.

PART 3: Wildebeest

15. Based on Figure 4, what kind of population growth model would you use to represent the Serengeti wildebeest population? Why?
The wildebeest population curve in Figure 4 appears to follow the shape of a logistic growth model. The population size increases initially, then it grows more slowly and eventually stabilizes at a constant value (the carrying capacity).

16. Was the wildebeest population at the carrying capacity in 1968? Why or why not?
No, the population was not at the carrying capacity in 1968 because it continued to grow for several years, until around 1980.

17. Calculate the size of the wildebeest population in the year 1968, using the logistic model simulator with the following settings: K = 1,245,000 wildebeest, r = 0.2717 per year, and N_0 = 80,000 wildebeest in the year 1958.
634,497 wildebeest

18. Imagine that the maximum per capita growth rate (r) for the wildebeest population increased to 0.4 per year in 1958.
    a. Suggest a specific reason that r could *increase* for a population.
    There are many possible reasons. Part 1 showed multiple factors that affect r*, including the birth rate,*

death rate, immigration rate, and emigration rate. An increase in the birth rate or immigration rate, or a decrease in the death rate or emigration rate, could all cause r to increase.

b. Recalculate the population size in 1968 using the new *r*. You can use the same values for the other settings as in Question 17.
982,852 wildebeest

c. Sketch or describe how the wildebeest population curve in Figure 4 might change as a result of the new *r*.
The population would grow more quickly and reach the carrying capacity sooner.

19. Imagine that the carrying capacity (*K*) for the wildebeest population decreased to 1,000,000 wildebeest in 1958.

a. Suggest a specific reason that *K* could *decrease* for a population.
There are many possible reasons. Any change that reduces how many wildebeest the environment supports could decrease K. *For example, the wildebeest's habitat could become smaller if part of the park is turned into cities or farmland, less rain could result in fewer plants for the wildebeest to eat, etc.*

b. Recalculate the population size in 1968 using the new *K*. You can use the same values for the other settings as in Question 17.
568,235 wildebeest

c. Sketch or describe how the wildebeest population curve in Figure 4 might change as a result of the new *K*.
The population would grow less quickly and stop growing once it reached the new, smaller carrying capacity.

20. Look at the size of the zebra population, which is shown as triangles in Figure 4, before and after the rinderpest vaccination campaign.

a. What patterns or trends do you observe in the zebra population?
The size of the zebra population stays fairly stable (does not change much) both during and after the rinderpest vaccination campaign.

b. Based on your answer above, what effect does rinderpest have on zebras?
It appears that zebras are not affected by rinderpest, since the decline in rinderpest had no effect on them.

21. Based on Figure 4, did the zebra population growth rate (*dN/dt*) differ in the years 1985 and 2003? Why or why not? (*Hint*: *dN/dt* at a given time is the slope of the population growth curve at that time.
The zebra population growth rate is given by the slope of the zebra population curve shown in Figure 4. This curve is fairly flat in both 1985 and 2003, suggesting that the zebra growth rate is nearly 0 in both years and does not differ much between years.

22. Imagine that there is a large wildfire in the Serengeti in 2010.

a. How might the zebra and wildebeest populations change right after the wildfire?
Fire would kill both wildebeest and zebra, so both populations would probably decrease in size.

b. How large do you think the zebra and wildebeest populations would be 50 years after the wildfire? Explain your answer, or what else you would want to know before making a prediction.
Over time, if the habitat was not severely damaged, the populations could return to the same carrying capacity as before the fire. Students might want to know more about how the fire affected the habitat or other factors that could affect the population size over the next 50 years.

23. We often design population models to answer certain questions. We may leave out other factors that are less relevant to our questions or that could overcomplicate our analysis.

a. Propose one *new* question about the waterbuck, kudu, or wildebeest populations that could be answered using the models you learned about in this activity.
Student answers will vary. Be open to a range of reasonable responses.

b. Propose one *new* question about the waterbuck, kudu, or wildebeest populations that could *not* be answered using these models. What could you add to the models in order to answer your question?
Student answers will vary. Be open to a range of reasonable responses.

CREDITS

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APPENDIX 1: Calculating $N(t)$ in the exponential growth model

The following equation describes the population growth rate (dN/dt) in the exponential growth model:

$$\frac{dN}{dt} = rN$$

Remember that N represents population size, t represents time, and r represents the maximum per capita growth rate. This equation is a first-order differential equation, which can be integrated to get an equation for N as a function of t.

First, separate the variables, N and t, by dividing both sides by N and multiplying by dt:

$$\frac{1}{N}dN = rdt$$

Next, integrate both sides and rearrange some terms:

$$\ln|N(t)| = rt + c$$
$$N(t) = e^{rt+c}$$

If we rewrite e^c as C:

$$N(t) = Ce^{rt}$$

From the equation above, we see that $N = C$ when $t = 0$. When $t = 0$, N also equals the initial population size N_0. So, C must equal N_0. Thus:

$$N(t) = N_0e^{rt}$$

APPENDIX 2: Calculating *N*(*t*) in the logistic growth model

The following equation describes the population growth rate (*dN/dt*) in the logistic growth model:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{1}$$

Remember that *N* represents population size, *t* represents time, *r* represents the maximum per capita growth rate, and *K* represents the carrying capacity.

Separate the variables, *N* and *t*, by dividing both sides by *N*(1–*N/K*) and multiplying by *dt*:

$$\frac{1}{N\left(1 - \frac{N}{K}\right)} dN = rdt \tag{2}$$

We need to integrate both sides of this equation to solve for *N*. However, the fraction on the left cannot be integrated directly, so we must first rewrite it using the method of partial fraction decomposition. Based on partial fraction decomposition:

$$\frac{1}{N\left(1 - \frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1 - \frac{N}{K}} \tag{3}$$

where *A* and *B* are unknown quantities that we must solve for. To do so, first multiply both sides by the denominator on the left:

$$\begin{aligned} 1 &= A\left(1 - \frac{N}{K}\right) + BN \\ &= A - \frac{AN}{K} + BN \end{aligned} \tag{4}$$

There are no terms containing *N* on the left, so the terms containing *N* on the right must cancel out. Thus:

$$\begin{aligned} \frac{-AN}{K} + BN &= 0 \\ BN &= \frac{AN}{K} \\ B &= \frac{A}{K} \end{aligned} \tag{5}$$

Substituting the value of *B* from (5) into (4) yields:

$$\begin{aligned} 1 &= A - \frac{AN}{K} + \frac{AN}{K} \\ &= A \end{aligned} \tag{6}$$

Substituting the value of *A* from (6) into (5), we find:

$$B = \frac{1}{K} \tag{7}$$

Substitute the value of *B* from (7) and the value of *A* from (6) back into (3):

$$\frac{1}{N\left(1 - \frac{N}{K}\right)} = \frac{1}{N} + \frac{\frac{1}{K}}{1 - \frac{N}{K}} \tag{8}$$

Substitute (8) into (2), integrate both sides, and rearrange the terms:

$$\int \left(\frac{1}{N} + \frac{\frac{1}{K}}{1 - \frac{N}{K}}\right) dN = \int rdt$$

$$\ln|N| - \ln\left|1 - \frac{N}{K}\right| = rt + c$$

$$ln\left|\frac{N}{1-\frac{N}{K}}\right| = rt + c$$

$$\frac{N}{1-\frac{N}{K}} = e^{rt+c} \tag{9}$$

If we rewrite e^c as C, (9) becomes:

$$\frac{N}{1-\frac{N}{K}} = Ce^{rt} \tag{10}$$

When $t = 0$, N equals the initial population size N_0. So, based on (10):

$$\frac{N_0}{1-\frac{N_0}{K}} = Ce^{0}$$

$$\frac{KN_o}{K-N_0} = C \tag{11}$$

Before substituting (11) into (10), let's rearrange (10) to get N by itself. First, multiply both sides of (10) by the denominator on the left:

$$N = \left(1-\frac{N}{K}\right)Ce^{rt} \tag{12}$$

Move all the terms containing N to the left side of the equation, then factor out N:

$$N + \frac{N}{K}Ce^{rt} = Ce^{rt}$$

$$N\left(1+\frac{Ce^{rt}}{K}\right) = Ce^{rt} \tag{13}$$

Rearrange (13) to get N on one side:

$$N = \frac{Ce^{rt}}{1+\frac{Ce^{rt}}{K}} \tag{14}$$

Substitute in (11) for C:

$$N = \frac{\frac{KN_o}{K-N_0}e^{rt}}{1+\frac{\frac{KN_o}{K-N_0}e^{rt}}{K}} \tag{15}$$

Multiply the numerator and denominator on the right side of (15) by $K - N_0$, then rearrange as follows:

$$N = \frac{KN_oe^{rt}}{(K-N_o)+\frac{KN_oe^{rt}}{K}}$$

$$= \frac{KN_oe^{rt}}{(K-N_o)+N_oe^{rt}}$$

$$= \frac{KN_o}{(K-N_o)e^{-rt}+N_o} \tag{16}$$

Rearrange the terms in the denominator to get the final equation for N as a function of t:

$$N(t) = \frac{KN_o}{N_o+(K-N_o)e^{-rt}} \tag{17}$$